Solutions about Regular Singular Points.

Again consider the homo^{n} linear diff^{n} eq^{n}

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0 \longrightarrow (1)$$

We assume that x_0 is a singular point of eq^{n} (1)

Then we are not assured of a power series sol^{n}

$$y = \sum_{n=0}^{\infty} c_n(x-x_0)^n \longrightarrow (2)$$

of (1) in powers of x_0.

An equation with a singular point x_0 does not, in general, have a solution of the form.

Then what type of solution is assured and what is the condition for that?

In order to answer the question let us first define what is regular singular point.

Definition.

Consider the differential eq^{n} (1) and its equivalent normalized form

$$y'' + P_1(x)y' + P_2(x)y = 0$$

$$\text{where } P_1(x) = \frac{a_1(x)}{a_0(x)} \text{ \& } P_2(x) = \frac{a_2(x)}{a_0(x)}$$

Also assume that at least one of the fns. $P_1(x)$ & $P_2(x)$ is not analytic at x_0, so that x_0 is a singular point of (1).

If the functions defined by the products

$$(x-x_0)P_1(x) \text{ and } (x-x_0)^2 P_2(x)$$

are both analytic at x_0, then x_0 is called a regular singular point of (1).

If either (or both) of the functions defined by the products is not analytic at x_0, then x_0 is called an irregular singular point.

Theorem.

Hypothesis : The point x_0 is a regular singular point of the diff^n eq^n (1)

Conclusion : The diff^n eq^n (1) has at least one non-trivial solution of the form

$$|x-x_0|^r \sum_{n=0}^{\infty} c_n (x-x_0)^n$$

where r is a definite (real or complex) constant which may be determined and this solution is valid in some deleted interval $0<|x-x_0|<R$ $(R>0)$ about x_0.

For example, $x=2$ is a regular singular point of the diff^n eq^n.

$$x^2(x-2)^2 y'' + 2(x-2)y' + (x+1)y = 0$$

[Here $P_1(x) = \frac{2}{x^2(x-2)}$ and $P_2(x) = \frac{x+1}{x^2(x-2)^2}$ so that

$(x-2)P_1(x) = \frac{2}{x^2}$ & $(x-2)^2 P_2(x) = \frac{x+1}{x^2}$ are both analytic at $x=2$]

Thus the eq^n has at least one non-trivial solution of the form

$$|x-2|^r \sum_{n=0}^{\infty} c_n (x-2)^n$$

valid in some deleted interval $0<|x-2|<R$ about $x=2$.

We also see that $x=0$ is also a singular point of the above eq^n but it is not a regular singular pt.. It is a irregular singular point (verify). And hence we can not ~~say~~ assured that the diff^n eq^n has a sol^n of the form

$$|x|^r \sum_{n=0}^{\infty} c_n x^n$$

in any deleted interval about $x=0$. It may or may not possess this type of solution.

Working Procedure:

Now we know that at least one solution is assured about a regular singular point x_0 of the differential equation (1).
But how do we proceed to find out the coefficients c_n and r in this solution?
The procedure through which we find these is known as method of Frobenius.

outline:

1) Let x_0 be a regular singular point of the diff^n eq^n (1) and assume a solution

$$y = (x-x_0)^r \sum_{n=0}^{\infty} c_n (x-x_0)^n, \quad c_0 \neq 0.$$

in some interval $0 < (x-x_0) < R$.

i.e, $$y = \sum_{n=0}^{\infty} c_n (x-x_0)^{n+r}.$$

2) Assuming term by term differentiation is valid

$$y' = \sum_{n=0}^{\infty} (n+r) c_n (x-x_0)^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n (x-x_0)^{n+r-2}$$

3) Substituting in the diff^n eq^n and simplify the resulting expression so that it takes the form

$$K_0 (x-x_0)^{r+k} + K_1 (x-x_0)^{r+k+1} + K_2 (x-x_0)^{r+k+2} + \cdots = 0$$

where k is a certain integer and the coefficients K_i $(i=0,1,2\ldots)$ are functions of r and certain of the coefficients c_n.

4. In order that the above expression be valid for all x in the deleted interval $0 < x - x_0 < R$, we must set

$$K_0 = K_1 = K_2 = \cdots = 0.$$

5. Upon equating to zero K_0 of the lowest power $r+K$ of $(x-x_0)$ we obtain a quadratic eq^n. in r which is called indicial equation of the diff^n eq^n.
Let the two roots of indicial eq^n be r_1 and r_2 where $Re(r_1) \geq Re(r_2)$.

6. Now equate all other remaining coefficients $K_1, K_2 \cdots$ to obtain a set of condition involving the constants r, which must be satisfied by the various coefficients c_n.

7. Now if r_1 and r_2 are real and unequal then choose the larger root r_1 (say) and substitute it into the above conditions and choose c_n to satisfy the condition.

8. If $r_2 \neq r_1$, we may repeat the above procedure with r_2, smaller root. But the solution here we gets may not be linearly independent of the ~~solution~~ previous solution if r_1 & r_2 are real and unequal.
Also if r_1 and r_2 are real and equal, then both the solutions are identical.

Thus in the general procedure discussed above it is indicated that two linearly independent solution are not guaranteed. Then it is natural to ask under what condition we are assured that the differential equation (1) has two linearly independent sol^n of the form $|x-x_0|^r \sum_{n=0}^{\infty} c_n (x-x_0)^n$ about a regular singular point x_0?

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Associated with this another question will crops up in mind that if the diff^n eq^n (1) does not have two linearly independent solution of the above form then what is the form of a solution that is linearly independent of the basic solution ?

To answer this question let us state the following theorem :

Theorem.

Hypothesis : Let the point x_0 be a regular singular pt. of the diff^n eq^n. Let r_1 and r_2 $(Re(r_1) > Re(r_2))$ be the roots of the indicial eq^n associated with x_0.

Conclusion

1. Suppose $r_1 - r_2 \neq N$ (N is a non-negative integer)
   Then the diff^n eq^n has two non-trivial linearly independent solutions y_1 & y_2 of form

$$y_1 = |x - x_0|^{r_1} \sum_{n=0}^{\infty} c_n (x - x_0)^n, \quad c_0 \neq 0$$

and

$$y_2 = |x - x_0|^{r_2} \sum_{n=0}^{\infty} c_n^* (x - x_0)^n, \quad c_0^* \neq 0.$$

2. Suppose $r_1 - r_2 = N$ (N is a positive integer).
   Then the diff^n eq^n (1) has two non-trivial linearly independent solution y_1 and y_2 given respectively by

$$y_1(x) = |x - x_0|^{r_1} \sum_{n=0}^{\infty} c_n (x - x_0)^n, \quad c_0 \neq 0$$

$$y_2(x) = |x - x_0|^{r_2} \sum_{n=0}^{\infty} c_n^* (x - x_0)^n + C y_1(x) \ln|x - x_0|$$

where $C_0^* \neq 0$ and C is a constant which may or may not be zero.

3. Suppose $r_1 - r_2 = 0$. Then the diff^n eq^n ① has two non-trivial linearly independent solution y_1 and y_2 given respectively by

$$y_1(x) = |x-x_0|^{r_1} \sum_{n=0}^{\infty} c_n (x-x_0)^n, \quad c_0 \neq 0$$

and $$y_2(x) = |x-x_0|^{r_1+1} \sum_{n=0}^{\infty} c_n^* (x-x_0)^n + y_1(x) \ln |x-x_0|$$

Ex.

Find power series sol^n of

$$2x^2 y'' + xy' + (x^2-3)y = 0 \longrightarrow ①$$

in some interval $0 < x < R$.

Sol^n:

We see that $x=0$ is a regular singular point of the given eq^n. (Here $P_1(x) = \frac{x}{2x^2}$, $P_2(x) = \frac{x^2-3}{2x^2}$
$P_1(x)$ & $P_2(x)$ are not analytic at $x=0$, so $x=0$ is singular pt.
Hence, $x P_1(x) = \frac{1}{2}$ and $x^2 P_2(x) = \frac{x^2-3}{2}$ are both analytic)

Let us assume one solution as $y = (x-0)^r \sum_{n=0}^{\infty} c_n (x-0)^n$, $c_0 \neq 0$

i.e., $$y = \sum_{n=0}^{\infty} c_n x^{n+r} \longrightarrow ②$$

Now differentiating term by term

$$y' = \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1}, \quad y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2} \longrightarrow ③$$

On substituting in the given eq^n and rearranging the terms in such a way that x in each term under summation will have same exponents we get

$$\sum_{n=0}^{\infty} [2(n+r)(n+r-1) + (n+r) - 3] c_n x^{n+r} + \sum_{n=2}^{\infty} c_{n-2} x^{n+r} = 0 \longrightarrow ④$$

or, $$[2r(r-1) + r - 3] c_0 x^r + [2(r+1)r + (r+1) - 3] c_1 x^{r+1}$$
$$+ \sum_{n=2}^{\infty} [\{2(n+r)(n+r-1) + (n+r) - 3\} c_n + c_{n-2}] x^{n+r} = 0 \longrightarrow ⑤$$

Equating the coefficients of lowest power of x we get the indicial eqn. as

$$2r(r-1)+r-3=0. \longrightarrow (6)$$

or $r = \frac{3}{2}, -1.$

let $r_1 = \frac{3}{2}$, $r_2 = -1$ $[\because r_1 > r_2]. \longrightarrow (7)$

Also $r_1 - r_2 = \frac{5}{2} \neq N$ (N is a non-negative integer)

Equating to zero the coeffts. of higher power of x we get $c_1[2(r+1)r + (r+1) - 3] = 0 \Rightarrow c_1 = 0. \longrightarrow (8)$

and $\{2(n+r)(n+r-1)+(n+r)-3\}c_n + c_{n-2} = 0.$ $[n \geq 2]$. $\longrightarrow (9)$

$\therefore n(2n+5)c_n + c_{n-2} = 0$ [choosing $r = r_1 = \frac{3}{2}$].

$$\therefore c_n = -\frac{c_{n-2}}{n(2n+5)}, \quad n \geq 2 \longrightarrow (10)$$

$\therefore c_2 = -\frac{c_0}{18}$, $c_3 = -\frac{c_1}{33} = 0$ $(\because c_1 = 0)$,

(All odd c_i's $= 0$ since $c_1 = 0$)

$c_4 = -\frac{c_2}{52} = \frac{c_0}{936}, \ldots\ldots$

Thus corresponding to the larger root $r = r_1 = \frac{3}{2}$ and using the values of $c_1, c_2, c_3 \ldots$ we obtain the solution as

$$y_1(x) = c_0 x^{3/2}\left(1 - \frac{1}{18}x^2 + \frac{1}{936}x^4 - \cdots\right) \longrightarrow (11)$$

Now let $r = r_2 = -1$ (smaller root)

We obtain $c_1 = 0$ and $n(2n-5)c_n + c_{n-2} = 0$, $n \geq 2$ (12)

[From (8) & (9)]

$$\therefore c_n = -\frac{c_{n-3}}{n(2n-5)} \longrightarrow (13)$$

$\therefore c_2 = \frac{c_0}{2}$, $c_3 = -\frac{4}{3} = 0$, $c_4 = -\frac{c_2}{12} = -\frac{c_0}{24} \ldots$

$$\therefore y_2(x) = c_0 x^{-1}\left(1 + \frac{1}{2}x^2 - \frac{1}{24}x^4 + \cdots\right)$$

Here $y_1(x)$ & $y_2(x)$ are linearly independent soln. So the general soln. is $y = c_1 y_1(x) + c_2 y_2(x)$ where c_1 & c_2 are arbitrary constant.